ON THE DIFFERENTIATION OF AN INFINITE SERIES TERM BY TERM.

By M. B. Porter.

If

$$f(x) = \sum_1^\infty u_n(x), \qquad a < x < b,$$

denote a convergent series of single valued, differentiable, real functions of the real variable x, the ordinary sufficient condition that the derivative $f'(x)$ exist and that

$$f'(x) = \sum_1^\infty u_n'(x),$$

namely, that each term $u_n(x)$ be differentiable and that the series $\sum_1^\infty u_n'(x)$ be uniformly convergent, is practically limited to the case where the uniformity can be tested by means of a so called M-series. That is, we shall have

$$f'(x) = \sum_1^\infty u_n'(x), \qquad a < x < b,$$

if there exist a convergent series of positive terms, $\sum_1^\infty M_n$, such that

$$|u_n'(x)| \leqq M_n, \qquad a < x < b. \tag{A}$$

The practical importance of this test in that, in particular, it is immediately applicable to power series and thus leads to a simple as well as rigorous treatment of undetermined coefficients, renders an elementary proof of it desirable. The following proof, while probably not new,* has not come to the notice of the writer elsewhere, and seems to be sufficiently elementary.

If x and $x + \Delta x$ lie in the interval of convergence of the series

$$f(x) = \sum_1^\infty u_n(x),$$

* Dini, *Functionen einer reellen Grösse*, p. 154, uses the Mean Value Theorem, but as he deduces a more general criterion, his proof is much less elementary.

then, by the primary definition of convergence, we have

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\sum_1^\infty \frac{u_n(x+\Delta x)-u_n(x)}{\Delta x},$$

and, by the first Mean Value Theorem,

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\sum_1^\infty u_n'(x+\theta_n\Delta x), \qquad 0<\theta_n<1. \tag{1}$$

If ϵ be an arbitrarily assigned positive number as small as we please, (A) tells us that a positive integer N can be found such that

$$\left|\sum_N^\infty u_n'(x+\theta_n\Delta x)\right|<\frac{\epsilon}{2}. \tag{2}$$

Furthermore, since each term $u_n(x)$ has a derivative $u_n'(x)$, a number δ can be found such that, if $|\Delta x|<\delta$,

$$\frac{u_n(x+\Delta x)-u_n(x)}{\Delta x}=u_n'(x+\theta_n\Delta x)$$

will differ from its limit, $u_n'(x)$, numerically by less than $\epsilon/2n$, so that

$$\left|\sum_1^n u_n'(x+\theta_n\Delta x)-\sum_1^n u_n'(x)\right|<\frac{\epsilon}{2}. \tag{3}$$

Hence it follows that

$$\left|\frac{f(x+\Delta x)-f(x)}{\Delta x}-\sum_1^\infty u_n'(x)\right|<\epsilon, \qquad |\Delta x|<\delta,$$

which proves the theorem.

The continuity, or even the integrability, of the derivatives $u_n'(x)$ is not postulated in the above proof, since the Mean Value Theorem merely requires that the derivatives exist.

The same proof can be applied when x denotes the complex variable, x_1+x_2i, the Mean Value Theorem* for functions of a complex variable now being employed.

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* Stolz, *Diff. u. Integralrechnung*, vol. 2, p. 95.